Homework 3 **Due 31.03.2009**

Please save your Mathematica code in a .nb file and e-mail it to me.

1.M (a) Show that the intersection of two principal ideals in $k[x_1, \ldots, x_n]$ is a principal ideal. In fact, show that if $I = \langle f \rangle$ and $J = \langle g \rangle$, then $I \cap J = \langle h \rangle$, where $h = LCM(f, g)$. (LCM=least common multiple=PPCM)

Thus computing the intersection of two principal ideals is easy if you can factor polynomials. In general, it is not always easy to factor polynomials over a given field k, and one has the following method.

[M](b) Suppose $I = \langle f_1, \ldots, f_r \rangle$ and $J = \langle g_1, \ldots, g_s \rangle$. Then $I \cap J = (tI + (1-t)J) \cap k[x_1, \ldots, x_n]$, where t is a variable different from x_i, and $(tI + (1-t)J)$ is the corresponding ideal. Thus in order to compute intersections of ideals one needs to consider the ideal

$$\langle tf_1, \ldots tf_r, (1-t)g_1, \ldots (1-t)g_s \rangle \subset k[x_1, \ldots, x_n, t],$$

compute a Groebner basis with respect to a lex order in which $t > x_i$ for all i, and then remove the polynomials which contain t. Write a program using Mathematica that computes the intersection of two ideals.

2 Show that $I = \langle xy, xz, yz \rangle$ is a radical ideal.

3 Show the following:

(a) $\langle x^2 + 1 \rangle$ is maximal in $\mathbb{R}[x]$.

(b) If $I \subset \mathbb{R}[x_1, \ldots, x_n]$ is maximal, show that $V(I)$ is either empty or a point in $\mathbb{R}^n$.

(c) Give an example of a maximal ideal I in $\mathbb{R}[x_1, \ldots, x_n]$ for which $V(I) = \emptyset$.

4. Prove that every prime ideal is radical.

5. Show that $I(\{(a_1, a_2, \ldots, a_n)\}) = \langle x_1 - a_1, \ldots, x_n - a_n \rangle$.